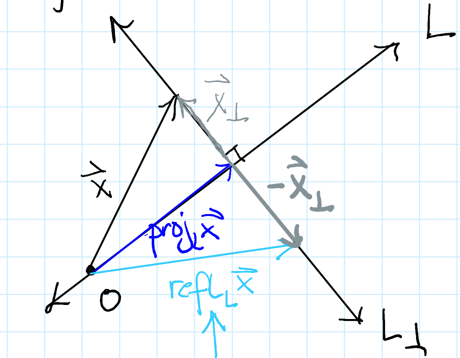
due Fri 11:59pm : HW3 $\longrightarrow$ Gradescope

HW4 becomes avail Sat 12:01am due following Fri

starting this weekend, you'll have (optional) practice problems on Lumen (zero cost homework platform)

§2.2 (conclusion)

Reflections




transformation : $T(\vec{x})$

reflect $\vec{x}$ about $L$

recall : $\text{proj}_L\vec{x} = \vec{x}_\parallel$

recall : $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$

$$\boxed{\vec{x} - \vec{x}_\parallel} = \vec{x}_\perp$$

$$T(\vec{x}) = \vec{x}_\parallel + (-\vec{x}_\perp)$$
$$= \vec{x}_\parallel - \boxed{\vec{x}_\perp} \quad \text{substitute}$$
$$= \vec{x}_\parallel - (\vec{x} - \vec{x}_\parallel)$$
$$= 2\boxed{\vec{x}_\parallel} - \vec{x}$$
$$= 2P\vec{x} - \vec{x}$$

recall : $\vec{x}_\parallel = \text{proj}_L\vec{x}$
$= \boxed{P\vec{x}}$

where $P = \begin{bmatrix} (u_1)^2 & u_1u_2 \\ u_1u_2 & (u_2)^2 \end{bmatrix}$

Know $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= 2\begin{bmatrix} (u_1)^2 & u_1u_2 \\ u_1u_2 & (u_2)^2 \end{bmatrix}\vec{x} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$$

$\uparrow$ $I_2$ ($\vec{x}$'s coefficient matrix)

$$= \left[2\begin{bmatrix} (u_1)^2 & u_1u_2 \\ u_1u_2 & (u_2)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\vec{x}$$

$$= \left[\begin{bmatrix} 2(u_1)^2 & 2u_1u_2 \\ 2u_1u_2 & 2(u_2)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]x$$

write as 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow$

$$= \begin{bmatrix} 2(u_1)^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2(u_2)^2 - 1 \end{bmatrix}\vec{x}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} & \\ 2u_1u_2 & 2(u_2)^2-1 \end{bmatrix}$$

where $b = c$

$$\downarrow$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \rightarrow \text{explore relationship between } a \text{ \& } d$$

$$a+d = 2(u_1)^2-1+2(u_2)^2-1$$
$$= \underbrace{2(u_1)^2+2(u_2)^2}\,-2$$
$$= 2\left(\cancel{(u_1)^2+(u_2)^2}\right)-2$$

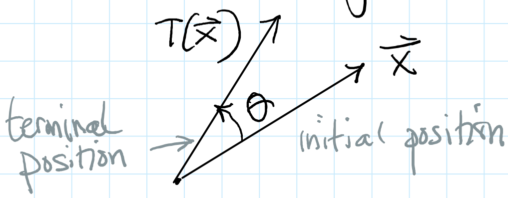
1 since $\vec{u}$ is unit vector $\|\vec{u}\| = 1$

$$a+d = 2-2$$
$$a+d = 0 \Rightarrow a=-d \Rightarrow d=-a$$

$\therefore$ can conclude that reflection matrix: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

## Rotations

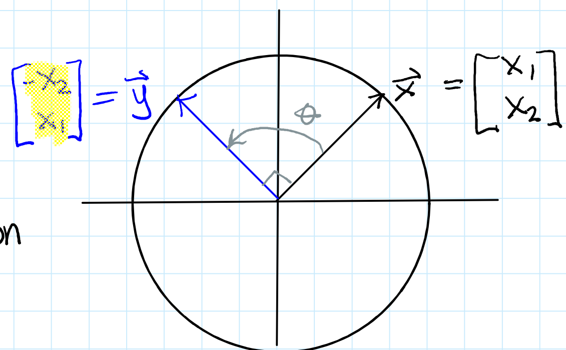
consider $\vec{x}$ rotated through a fixed angle, $\theta$, CCW
result would be a linear transformation, $\mathbb{R}^2 \to \mathbb{R}^2$




Let $\theta = \frac{\pi}{2}$

$\downarrow$

$90^\circ$ CCW rotation



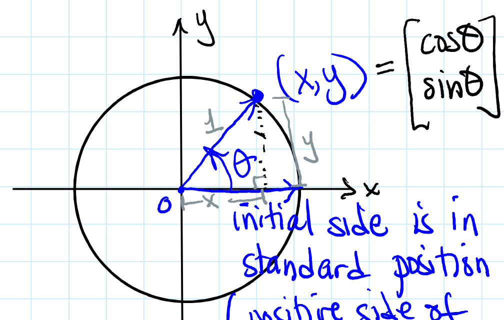

$\vec{x}$ initial position

$\vec{y}$ terminal position

Generalize for any $\theta$:

$$T(\vec{x}) = \cos\theta\,\vec{x} + \sin\theta\,\vec{y}$$
$$= \cos\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sin\theta \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\, x_1 \\ \cos\theta\, x_2 \end{bmatrix} + \begin{bmatrix} -\sin\theta\, x_2 \\ \sin\theta\, x_1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \cos\theta\, x_2 + \sin\theta\, x_1 \end{bmatrix}$$

reorder terms




initial side is in standard position (positive side of x-axis)

$$\begin{bmatrix} \cos\theta\, x_2 + \sin\theta\, x_1 \end{bmatrix}$$ reorder terms

$$= \begin{bmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

can write as

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is rotation matrix

ex. find the matrix of a CCW rotation through $\frac{\pi}{6}$

$$\theta = \frac{\pi}{6} \qquad \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta = \frac{\pi}{6} = 30^\circ$$

$$= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\pi/6 & \cos\frac{\pi}{6} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \quad \text{OR} \quad \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

## § 2.3 Matrix Products

recall composition: $y = \sin x$
$z = \cos y$

$$\begin{tikzcd} z & \\ & y \arrow[ul, "z=\cos y"'] \\ x \arrow[uu, "z=\cos(\sin x)"] \arrow[ur, "y=\sin x"'] & \end{tikzcd}$$

i.e., how write $z$ ITO $x$

can also compose transformations

recall: actual vs. encoded position example

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ given $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

encoding matrix

encoding transformation: $\vec{y} = A\vec{x}$

suppose there's an intermediate location where a separate encoding is done

$\vec{z} = B\vec{y}$ given $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

would be found using

$\vec{z} = B\left(A\vec{x}\right)$

$$\begin{tikzcd}
\vec{z} & & \vec{y} \arrow[ll, "\vec{z} = B\vec{y}\text{, } B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}"'] \\
& \vec{x} \arrow[ul] \arrow[ur, "\vec{y} = A\vec{x}\text{, } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}"'] &
\end{tikzcd}$$

$\vec{x}$ (boat)

show that $\vec{z} = T(\vec{x})$ is a <u>linear</u> transformation:

$\vec{y} = A\vec{x}$ $\quad\quad$ $\vec{z} = B\vec{y}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad\quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

can be written as:

$$y_1 = x_1 + 2x_2 \quad\quad z_1 = 6y_1 + 7y_2$$

$$y_2 = 3x_1 + 5x_2 \quad\quad z_2 = 8y_1 + 9y_2$$

write $\vec{z}$ ITO $\vec{x}$

$$z_1 = 6(x_1 + 2x_2) + 7(3x_1 + 5x_2) \quad\quad z_2 = 8(x_1 + 2x_2) + 9(3x_1 + 5x_2)$$

$$= 6x_1 + 12x_2 + 21x_1 + 35x_2 \quad\quad = 8x_1 + 16x_2 + 27x_1 + 45x_2$$

$$z_1 = 27x_1 + 49x_2 \quad\quad z_2 = 35x_1 + 61x_2$$

<u>is</u> a <u>linear</u> transformation since could write $z_1, z_2$ as <u>linear</u> equations

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 27 & 49 \\ 35 & 61 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

composed transformation matrix

look at it a different way:

recall property: $\boxed{T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})}$

then $\quad B\left(A\left[\vec{v} + \vec{w}\right]\right)$

$$= B\left(A(\vec{v}) + A(\vec{w})\right)$$

$$= B(A\vec{v}) + B(A\vec{w}) \quad\quad \text{property holds}$$

$= B(A(\vec{v}) + A(\vec{w}))$

$= B(A\vec{v}) + B(A\vec{w})$ property holds

likewise, know $T(k\vec{v}) = kT(\vec{v})$

then $B(A(k\vec{v}))$

$= B(k(A\vec{v}))$

$= kB(A\vec{v})$ property holds